On the Prospects of Chaos Aware Traffic Modeling

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Abstract— **In this paper the chaotic properties of the TCP congestion avoidance mechanism are investigated. The analysis focuses on the origin of the complex behavior appearing in deterministic TCP/IP networks. From the traffic modeling point of view the understanding of the mechanism generating chaos is essential, since present models are unable to cope with this phenomena.**

Using the basic tools of chaos theory in our study, the main characteristics of chaotic dynamics are revealed. The dynamics of packet loss events is studied by a simple symbolic description. The cellular structure of the phase space of congestion windows is shown. This implies periodic behavior for large time scales.

Chaotic behavior in short time scales and periodicity for larger times makes it necessary to develop models that account for both. Thus a simple model that describes the congestion window dynamics according to fluid equations, but handles the packet loss events separately is introduced. This model can reproduce the basic features observed in realistic packet level simulations.

I. INTRODUCTION

Recently Veres and Boda [1] have demonstrated that TCP congestion control can be chaotic in certain circumstances. Chaos in practice means that TCPs influencing each other in a computer network can produce highly complex behavior in time which is sensitive to small perturbations, yet the equations describing it are deterministic and simple. Understanding the mechanism and equations that produce chaos in a TCP/IP network is crucial in traffic modeling. If we can identify the details of the mechanism then it is possible to build new kinds of network traffic models where the factors important from the point of view of dynamics and chaos are kept while many unimportant factors are reduced or simplified. Current TCP models are either too simplistic by assuming periodic behavior or they are entirely stochastic disregarding small details of a given network which might alter the dynamics completely and even change its statistical properties.

Stochastic models can also break the temporal and spatial (topological) structure of correlations existing among TCPs in an extended network and are not able to give a correct account of possible long range dependence born between far away TCPs [2]. On the other hand, new chaos aware TCP and network models can preserve correlations and show the same level of complexity as it is observed in real network traffic. These features might turn out to be unavoidable when building reliable models of large networks where packet level simulation requires astronomical computer resources and/or simulation time. Moreover, chaos theory itself provides new tools to quantify this complexity. The complexity of a real network traffic or its packet level simulation and the complexity of a traffic model can be compared. Despite their inherent simplicity, these tools have never been used in networking and they might open new prospects in verifying TCP and network models or can even characterize the actual network state.

We think that all these issues are so important, that we should answer at least the basic open questions concerning the chaotic state of the TCP congestion control mechanism. For example we still do not know what causes TCP chaos exactly. Accordingly we do not know how generic its appearance is. For example, it has been argued in Ref. [3] that in the chaotic examples shown in Ref. [1] the packet loss probability is considerably higher than 1%. It has been shown [3–5] that in this case the exponential backoff mechanism plays an important role and can be responsible for the complex congestion window dynamics. We are going to show that chaos is not a consequence of high network congestion or loss and that TCPs operating in congestion avoidance mode, never entering into a backoff state show chaos. In other words chaos is the generic behavior of many TCP systems and periodicity and synchronization is rather exceptional.

Also we have to ask how we can distinguish TCP chaos from stochasticity and do we gain anything by doing that.

To point out the main weakness of stochastic models and to call for chaos aware models we would like to show that current stochastic TCP models are even unable to predict the traffic in a simple scenario like the one shown in Fig. 1 and investigated throughout this paper. In this setup
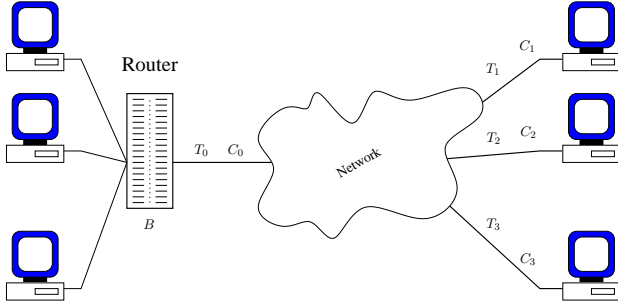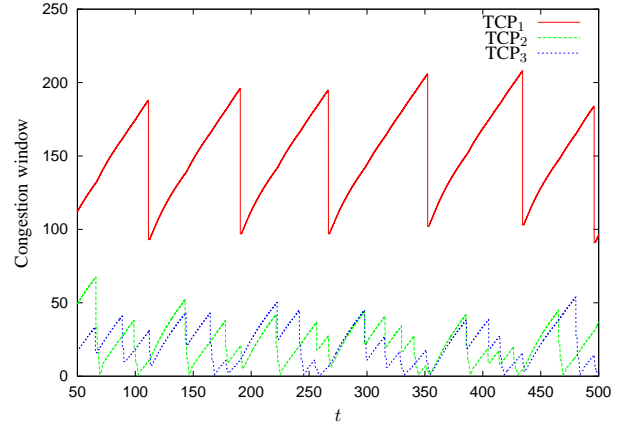

Fig. 1

INVESTIGATED NETWORK MODEL: THERE ARE THREE TCP FLOWS SHARING A COMMON BUFFER THAT CAN STORE B PACKETS AND A COMMON LINE WITH DELAY D_0 AND SPEED C_0. THEN THE COMMON LINK SPLITS INTO THREE DIFFERENT LINES WITH DIFFERENT DELAYS AND SPEEDS. IN THE ACTUAL SIMULATIONS $B = 100$, $T_0 = 400 \,\mathrm{ms}$, $T_1 = 100 \,\mathrm{ms}$, $T_2 = 150 \,\mathrm{ms}$, $T_3 = 200 \,\mathrm{ms}$, $C_0 = 10^6 \,\mathrm{bps}$ AND $C_1 = C_2 = C_3 = 10^7 \,\mathrm{bps}$ HAS BEEN CHOSEN.

three TCP flows sharing a common buffer that can store B packets and a common line with delay T_0 and speed C_0, then splitting into three different lines with different delays and speeds. In the actual simulations $B = 100$, $T_0 = 400 \,\mathrm{ms}$, $T_1 = 100 \,\mathrm{ms}$, $T_2 = 150 \,\mathrm{ms}$, $T_3 = 200 \,\mathrm{ms}$, $T_0 = 10^6 \,\mathrm{bps}$ and $C_1 = C_2 = C_3 = 10^7 \,\mathrm{bps}$ has been chosen. The congestion windows of the competing TCPs are not limited by the senders or by the receivers. The injected TCP packets can be lost only at the bottleneck buffer, there are no random losses on the links or in other buffers. It follows that the traffic is controlled by strict deterministic rules, that is by the TCP Reno algorithm [6]. Numerical simulations were carried out by Network Simulator (`ns`) version 2b5 [7].

Based on ideas brought from stochastic TCP modeling a common belief is that TCP is biased against long round trip time connections and the throughputs are proportional to $\sim 1/T_{\mathrm{RTT}}^2$. This assumption has been proven to be very good in the presence of random elements in the simulation [8]. In reality, shown in Fig. 2(a) it can be observed that the congestion window corresponding to the *largest* round trip time is significantly larger than the others. This TCP obtains *unfairly* higher throughput than the other two. The packet loss rate of the preferred TCP ($\sim 5 \cdot 10^{-5}$) is also an order of magnitude less than that of the suppressed ones

($\sim 7 \cdot 10^{-4}$). The "winner" TCP behaves seemingly periodically, while the "losers" seem to be erratic. Obviously,

(a)
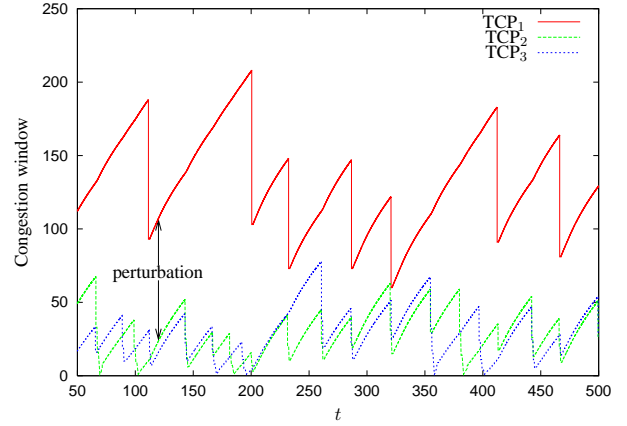
(b)

Fig. 2

(A) A TYPICAL PART OF THE CONGESTION WINDOW TIME SERIES IS SHOWN FOR THE SIMULATION OF THE SCENARIO OF FIG.1. (B) THE EFFECT OF A SMALL PERTURBATION AT $t = 120 \,\mathrm{s}$ ON THE CONGESTION WINDOW DEVELOPMENT.

there is something here which is missed completely by the stochastic model. Next, we show that this simple scenario is already chaotic and the deterministic nature of packet losses cannot be disregarded if we would like to build models that correctly predict the temporal behavior of congestion windows.

II. THE TCP BUTTERFLY EFFECT

The complete state of a TCP can be given by a number of internal variables at any moment [6]. Such variables

are the congestion window, the slow start threshold, the retransmission timeout, the backoff counter, the duplicate ACK counter, and so on. However, during the optimal operation of TCP, in congestion avoidance mode, a single variable can be selected which controls almost completely the behavior of the TCP: that is the congestion window. This variable has also practical importance, since it limits the maximum number of unacknowledged packets sent by a TCP into the network.

To demonstrate how *sensitive* this system can be for small perturbations the same simulation has been run for $120\,\mathrm{s}$ and then a perturbation of $\delta w_i(0) = 0.01$ has been added to all the congestion window values at $t = 120\,\mathrm{s}$. The result is shown in Fig. 2(b). The congestion windows remain unchanged until $t = 120\,\mathrm{s}$. Then the difference between the congestion windows of the two simulations $|\delta\mathbf{w}(\tau)|$ remains the same (~ 0.01) until the first packet drop event. Then one of the underprivileged TCPs, whose packet has been lost, halves its window. As a result, the distance between the original and the perturbed trajectories grows about an order of magnitude, since the owner of the lost packet differs in the original and in the perturbed simulations. At this point it seems that the dominant TCP is not affected at all. Finally, around $t = 240\,\mathrm{s}$ the time evolution of the dominant TCP diverges completely from the original trace due to a permutation of packets, resulting in a loss event for the dominant TCP. As we can see the rest of the simulation differs from the original one. Such sensitivity against small perturbations is called the butterfly effect in chaos theory and gives us the first clue that this system operating in congestion avoidance is actually chaotic. Next we introduce a few basic tools which help us to characterize the chaotic state.

III. CHARACTERIZING CHAOS

One of the most basic tools of chaos theory in visualizing the dynamics is the Poincaré surface of section. Instead of looking at the continuous time evolution of trajectories one can select a surface in the phase space and watch only when the trajectories cross that surface. In case of TCP congestion window trajectories the evolution between two packet loss events is fairly simple. All the interesting things happen at the times of packet losses. Therefore the values of congestion windows taken at the moments of packet losses of any of the TCPs is a natural choice for a surface of section in general. In Fig. 3 this surface of section is shown for our system. One can see that the congestion window triplets (w_1, w_2, w_3) taken at times of packet losses approximately form a two dimensional surface within the three dimensional congestion window phase space. It is
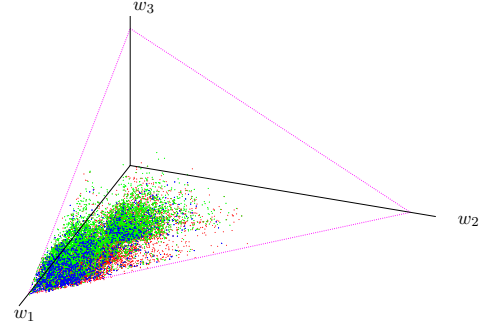

Fig. 3

easy to understand why we get such a surface: packet losses occur when the buffer is full. In the scenario of Fig. 1 the packets first fill the lines denoted by 0, 1, 2 and 3. The number of packets which can travel on these lines is given by the bandwidth delay products divided by the packet size $C_0 T_i/P$, where the packet size in our simulations was 512 bytes. The maximum number of packets on the lines and in the buffer might be approximated by $Q = B + C_0 T_0/P + \frac{1}{3} C_0 (T_1 + T_2 + T_3)/P$, as all packets go through link 0, while each packet should choose either one of the three lines 1, 2 or 3. The sum of congestion windows $W = \sum_i w_i$ is approximately the number of packets in the network and packet loss occurs approximately when $W = Q$. This equation defines the "surface of loss" inside the window phase space. This surface is also indicated in Fig. 3.

In chaotic systems the attractor is often a fractal object. Fractals are statistically self-similar geometric objects that might be characterized by suitably defined non-integer valued dimensions. For ordinary fractals this dimension is less then the Eucledian embedding dimension D of the object [9]. The fractal dimension of the points on the surface of section can be measured. To do this we can project the points onto a suitable surface. The points were projected onto the $\sum w_i = const.$ surface and the fractal dimension of this two dimensional projection was measured. A usual method for measuring the fractal dimension is when a grid of cells of size ϵ is put on the ob-

ject, and the number of non-empty cells $N(\epsilon)$ is counted. The *box counting dimension* is given by

$$D_0 = -\lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon}. \tag{1}$$

In Fig. 4 it can be seen that the points on the surface form a non-trivial fractal with box counting dimension $D_0 = 1.69 \pm 0.02$.
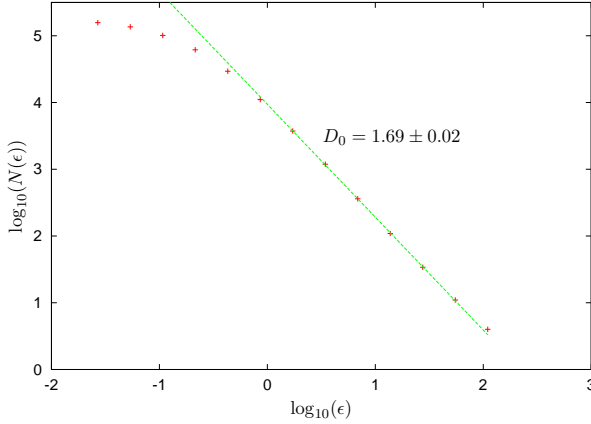

Fig. 4

FRACTAL DIMENSION OF THE TWO DIMENSIONAL PROJECTION OF THE SURFACE OF SECTION OF FIG. 3.

Now our qualitative picture of TCP dynamics in congestion avoidance mode can be summarized as follows. Congestion windows steadily grow between packet losses. This process can usually be well approximated [10, 11] with fluid equations of the type

$$\frac{dw_i(t)}{dt} = \frac{1}{T_{\text{RTT},i}(\mathbf{w})}, \tag{2}$$

where $T_{\text{RTT},i}(\mathbf{w})$ is the round trip time of the ith TCP. Round trip times can also be approximated as functions of the congestion windows and then the resulting differential equations (2) can be solved self-consistently. The sum of congestion windows also grows steadily and the congestion windows cut the surface of loss at some point. Then one of the congestion windows is halved according to the TCP algorithm and the position of the point in the phase space drops below the loss surface and the process starts again.

IV. SYMBOLIC DESCRIPTION

One can see that the dynamical process between packet losses described above is relatively simple. The complicated fractal structure of the attractor and the sensitivity for perturbations should come from the fine details of the packet loss process. Each time the congestion window trajectory crosses the loss surface a packet loss event happens in one of the TCP flows. One of the symbols $S_i = \{1, 2, 3\}$ can be assigned to the ith packet loss depending on which TCP lost the packet. We can consider the sequence of the symbols $\ldots S_{i-1} S_i S_{i+1} \ldots$ generated by the time evolution of our TCPs. This symbol sequence is a coding of the real congestion window evolution. Such symbolic coding plays an important role in the theoretical description of chaotic systems. When an infinite sequence of symbols codes exactly one or zero real space trajectory the coding is called *Markov partition*. In this case the symbol sequences uniquely code the chaotic dynamics. An equivalent definition of the Markov partition is when each *periodic* symbol sequence codes exactly one or zero *periodic* orbit of the chaotic system.

In the case of TCP congestion avoidance mode we can demonstrate that the introduced symbols form a Markov partition. If an infinite periodic sequence (such as $\ldots 122312231223 \ldots$) is prescribed and two different initial congestion window triplets $\big((w_1, w_2, w_3)$ and $(w_1', w_2', w_3')\big)$ are taken and they evolve according to the equations (2), they will reach the loss surface at different points. One can prove that the equations (2) are linearly stable, so they are not capable to increase the difference between two orbits. When one of the windows is halved after the trajectory crosses the loss surface, then the difference between the orbits is halved in that direction while it remains unchanged in other directions. The time evolution according to Eq. (2) and the prescribed halvings will decrease the distance between the two orbits in each period. As all the TCPs should halve their windows at least once in each cycle the distance between the two initial triplets is at least halved in each period. This way we can see that two trajectories started from different initial conditions converge exponentially to a common periodic orbit. In the end we get a unique periodic orbit corresponding to a given periodic code. So far we forced a given TCP to halve its window according to the prescribed symbol. In the end we can look at the actual packet flow generated by the window evolution. The prescribed periodic orbit can be feasible if the resulting packet flow is in accordance with the prescribed packet loss sequence or it is not feasible if the resulting packet flow generates losses in a different order than it has been assumed. This way we can decide if the calculated periodic orbit exists or not. This procedure ensures that we assign one or zero real periodic orbits to a periodic sequence and proves the existence of the Markov partition.

V. THE TOOL-BOX OF CHAOS

The statistical theory of chaos [12] is based on the symbol sequences introduced above. If the dynamics is regular (non-chaotic) then the dynamics is periodic or quasi periodic while the most important characteristics of chaos is that it endlessly generates topologically different new trajectories. This is reflected in the way different systems generate symbol sequences. A length n symbol sequence can continue in many ways; in average with a number of symbols to form a length $n + 1$ sequence. Thus the number of length $n + 1$ sequences $N(n + 1)$ can be expressed as

$$N(n + 1) \approx aN(n), \tag{3}$$

when the length n is large. In chaotic systems there is more than one possibility to continue a sequence in average and $a > 1$ while in regular systems $a = 1$ for long sequences. Consequently in chaotic systems the number of possible length n symbols grows exponentially

$$N(n) \sim a^n \sim e^{K_0 n}, \tag{4}$$

and the quantity $K_0 = \ln(a) > 0$ is the *topological entropy*. In non-chaotic systems the number of sequences grows sub-exponentially and the topological entropy is zero.

To prove that our TCP system really produces chaos we can measure its topological entropy. In our case we can have a maximum of 3^n different symbolic sequences of length n. Of course not all of them are realized by the dynamics since some of them are impossible. For example infinite sequences consisting of only one or two symbols are excluded since this would imply that some of the congestion windows are never halved. In Fig. 5 we show the number of realized symbol sequences for different lengths n up to $n = 12$. The measurement has been carried out by logging out the symbols (i.e. the index of the TCP which lost a packet) from an `ns` simulation of the system. We generated a sequence of 150.000 consecutive symbols and determined how many different length n sequences exist in it.

We observed that in average $a = 2.586$ symbols can follow a given symbol sequence and the topological entropy is $K_0 \approx 0.953$. This shows that our system is strongly chaotic as the number of sequences grows with a large exponent, yet it is markedly different from a stochastic system, where all combination of symbols are allowed and would result in $a = 3$ and a topological entropy of $\ln 3$.

The procedure described so far measures the existence of different symbol sequences only. We can characterize chaos further by calculating also the probability
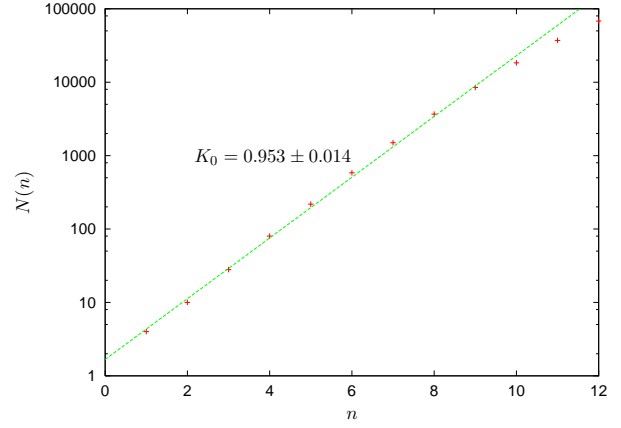
Fig. 5

NUMBER OF DIFFERENT SYMBOLS AS A FUNCTION OF SYMBOL LENGTH FOR OUR SYSTEM ON A SEMI-LOGARITHMIC PLOT. THE FITTED LINE IS $N(n) = 1.669 \times 2.586^n$

$P(S_1, S_2, \dots, S_n)$ of the occurrence of the symbol sequence $S_1, S_2, \dots, S_n$. In a system with L symbols ($S_i = \{1, 2, \dots, L\}$) we can visualize this probability distribution by assigning the number

$$x = \sum_{i=1}^{n} (S_i - 1) L^{-i} \tag{5}$$

to each symbol sequence and plotting $P(x)$. In fact $0 \le x < 1$ is the L-ary fractional representation of the number represented by the symbols.

In our system we carried out this analysis and the result is plotted in Fig. 6. It can be clearly seen that the
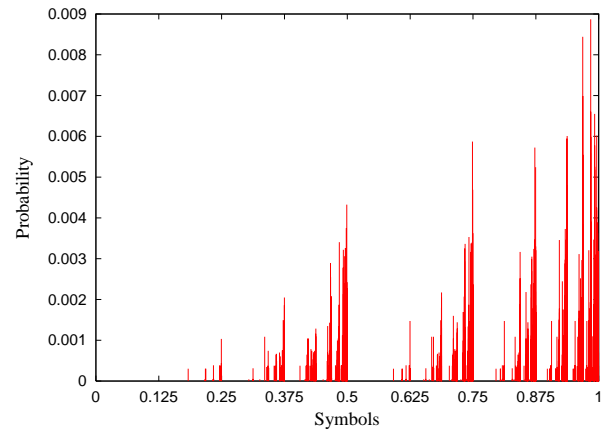
Fig. 6

PROBABILITY DISTRIBUTION OF SYMBOLS OF LENGTH 10 FOR OUR SYSTEM.

probability distribution is a fractal in the space of sym-

bols. In fact, in all chaotic systems we should observe a multifractal distribution and the topological entropy introduced above is related to the box counting dimension of this representation. If we cut the $[0,1[$ interval into boxes of size $\epsilon = 1/L^n$ then the number of non-empty boxes is the number of existing symbols $N(n)$. The box counting dimension is then

$$D_0 = \lim_{n\to\infty} \log N(n)/\log L^n = K_0/\ln L.$$

In our case then the box counting dimension of the multifractal of Fig. 6 is 0.864. Note, that this box counting dimension is *not* related to the box counting dimension of the attractor discussed before.

Scaling of moments of the probability distribution give further characterization of the multifractal properties in the symbol space. We can define the Rényi entropies [13]:

$$K_q = \lim_{n\to\infty} \frac{1}{n}\frac{1}{1-q} \ln \sum_{\{S\}_n} P^q(S_1, S_2, \dots, S_n), \quad (6)$$

where summation $\{S\}_n$ goes over all possible symbol sequences of length n. The quantity $K_q/\ln L$ again measures the D_q generalized dimension of the multifractal spectrum of the $P(x)$ histogram. The most important entropy is the Kolmogorov–Sinai entropy $K_1 = \lim_{q\to 1} K_q$ which gives the scaling of the Shannon entropy of the probability distribution:

$$K_1(n) = - \sum_{\{S\}_n} P(S_1, S_2, \dots, S_n) \ln P(S_1, S_2, \dots, S_n),$$
$$(7)$$

$$K_1 = \lim_{n\to\infty} \frac{1}{n} K_1(n). \quad (8)$$

In Fig. 7 the Kolmogorov–Sinai entropy is measured for our system.

The Kolmogorov–Sinai (KS) entropy in a chaotic system is also related to the Lyapunov exponent λ of the mapping of the Poincaré section onto itself. If we consider two nearby trajectories on the Poincaré section—which is the loss surface in our case—then their initial separation in the phase space δw_0 grows each time the trajectories revisit the section. After n revisits the distance grows exponentially $\delta w_n \approx e^{\lambda n} \delta w_0$ where the average of the exponent lambda $\langle \lambda \rangle$ is the Lyapunov exponent of the Poincaré section. The KS entropy gives the Lyapunov exponent $K_1 = \langle \lambda \rangle$ in our system. The positivity of the KS entropy is another indication of chaos and the exponential sensitivity for the perturbation of trajectories.

Since the structure of the Markov partition for TCPs is simple even for larger networks, the topological and KS
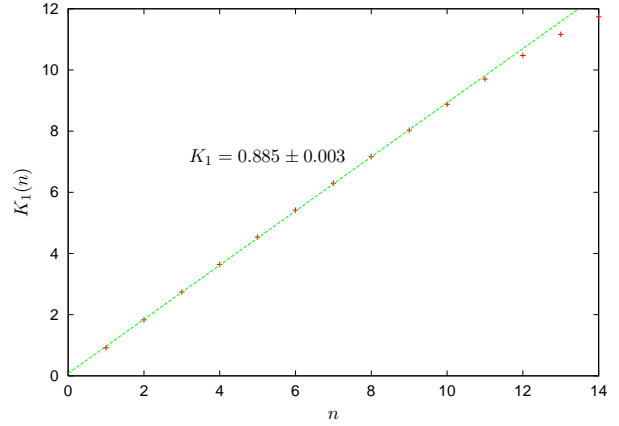

Fig. 7

KOLMOGOROV–SINAI ENTROPY FOR OUR SYSTEM.
$K_1(n) = - \sum_{i=1}^{3^n} p_i \ln p_i = 0.885 \times n + 0.078$

entropies are easy to measure in a simulation or can even be determined in a real network. These quantities measure the complexity of the dynamics and by evaluating them we can quantify complexity and evaluate TCP and network models in general.

VI. CELLULAR CHAOS

The results so far confirmed the hypothesis of chaotic dynamics. However, the variables of TCP in reality are discrete and not continuous. We can demonstrate this by applying such a small perturbation to congestion windows by which we do not change the loss events happening in the system. To investigate this we perturbed the trajectory of Fig. 2(a) by a small vector $\delta \mathbf{w}$ and traced the difference between the original and the perturbed trajectory. We found that there exists a set of perturbations, shown in Fig. 8, which vanishes after all windows are halved. The maximal perturbation of this type was found to be approximately $|\delta \mathbf{w}| < 0.001$. There is a well defined neighborhood around every phase point which defines the the same behavior of loss dynamics. That is, after all the congestion windows are halved, the time evolution of congestion windows becomes identical, mainly because the algorithm sets the window to its integer part. This means that the phase space is not continuous. It is divided into small "attractive cells" in which trajectories converge in finite time. If a trajectory gets into the cell of another trajectory then they follow the same trajectory later on.

This property makes TCP chaos very interesting from a theoretical point of view, since we have a globally chaotic dynamics while the fine details of the system are non-chaotic. This is not typical in natural occurrences of chaos
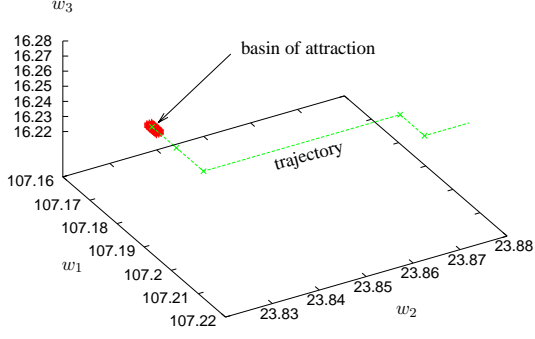
Fig. 8

OUR TRIAL TCP TRAJECTORY (GREEN) AND THE POINTS IN
PHASE SPACE (RED) WHICH FOLLOW THE SAME TRAJECTORY
AFTER ALL THREE WINDOWS ARE HALVED. THE BASIN OF
ATTRACTION OF THE STARTING POINT OF THE TRAJECTORY IS
APPROXIMATELY A CUBE OF SIZE 0.001^3.

but it is potentially important in engineered systems. The most significant aspect of the cellular structure is that all trajectories should be periodic. If a trajectory re-enters a cell visited previously then according to the attractive nature of the dynamics it will follow the same trajectory again and will repeat itself. Since the phase space can be divided into a finite number of cells a trajectory should at least repeat itself after visiting all the possible cells. In reality the repetition of a cell happens much earlier. We can make an estimate of the typical length and the distribution of the periods of trajectories following the theory developed by Grebogi, Ott, and Yorke [14] for chaotic systems with numerical roundoff.

First let us consider the mapping connecting two consecutive points on the loss surface. Also we can divide the loss surface into N attractive cells. Window values falling into the same cell will follow the same trajectory later on. Now suppose that we iterate the mapping n times, and the orbit visited none of the cells twice. At this point we choose a cell random from the n visited one, and try to calculate the probability that in step $n+1$ the system visits the chosen cell. The following calculations assume that the dynamics of the map is mixing (see [14] for details). The result of mixing is that the typical value of $n - j$ is large, where j is the iterate when the chosen cell was first visited. Thus $\bar{l}$ becomes large, where $l = n - j + 1$, and bar denotes expectation value. Now the probability of repeating the chosen cell in step $n+1$ is equal to the probability of visiting that cell. On the other hand, due to random initial conditions, the probability that a given cell is visited

in step j is also equal to the probability of the attractor to visit that cell. Combining these, for the probability of repeating the cell in step $n+1$ that was first visited in step j one obtains

$$\langle p \rangle = \sum_i p_i^2, \tag{9}$$

where p_i is the probability with which the orbit visits the ith cell (i.e. the measure of the attractor in that cell). So the probability of repeating any of the previous cells in step $n+1$, supposing that the orbit did not repeat during the previous n steps is:

$$p_{\text{rep}}(n) = n\langle p \rangle. \tag{10}$$

Thus the probability that an orbit of length n has no repeats is

$$p_{\text{norep}}(n) = \prod_{k=1}^{n-1}(1 - p_{\text{rep}}(k)) = \prod_{k=1}^{n-1}(1 - k\langle p \rangle). \tag{11}$$

One can approximate $\ln p_{\text{norep}}(n)$ for $n\langle p \rangle \ll 1$:

$$\ln p_{\text{norep}}(n) = \sum_{k=1}^{n-1}\ln(1 - k\langle p \rangle)$$
$$\approx -\sum_{k=1}^{n-1}k\langle p \rangle \approx -n^2\langle p \rangle/2, \tag{12}$$

which yields

$$p_{\text{norep}}(n) \approx \exp[-(n^2\langle p \rangle/2)]. \tag{13}$$

The condition for this approximation ($n\langle p \rangle \ll 1$) is justified if $\bar{l}\langle p \rangle \ll 1$. As we shall see, $\bar{l} \sim \langle p \rangle^{1/2}$, thus the above condition is equivalent to $\bar{l} \gg 1$. The probability that an orbit goes n steps without repeat and then repeats on step $(n+1)$ is

$$p(n) = p_{\text{rep}}(n)\,p_{\text{norep}}(n). \tag{14}$$

If we assume that the probability of returning to any of the preceding cells is equal, then if the first repeat occurs at step $n+1$, the probability that $n - j + 1 = l$ is

$$p_{\text{p}}(n, l) = \begin{cases} 1/n & \text{for } l \leq n, \\ 0 & \text{for } l > n. \end{cases} \tag{15}$$

For large l the probability density of l becomes

$$P(l) = \int_0^\infty p(n)\,p_{\text{p}}(n, l)\,dn$$
$$= \frac{\sqrt{8}}{\pi}\langle p \rangle^{1/2} F\left(\langle p \rangle^{1/2} l\right), \tag{16}$$

where we substituted the results above, and where

$$F(y) = \sqrt{\pi/8} \int_y^\infty \exp(-x^2/2)\, dx.$$

Calculating the expectation value of l, we obtain the average length of periodic orbits generated by the discrete, cellular structure of the phase space:

$$\bar{l} = \int_0^\infty l\, P(l)\, dl = \sqrt{\pi/8}\, \langle p \rangle^{-1/2}. \qquad (17)$$

In the next section we investigate the implications of the periodicity of the trajectories.

VII. EXPLORING PERIODIC ORBITS

Until now three parallel TCPs were studied in our simulation scenario. In such situation the number of cells which can be distinguished from each other in the phase space is in the order of $\sim 10^{15}$. Also, the length of a typical periodic orbit is enormous and we do not expect to observe them in realistic simulations.

Therefore, it seems more reasonable to find periodic orbits when only two TCPs are operating in the network. For this scenario we used the same network model that we have introduced in Fig. 1 except that the TCP with the largest round-trip time was removed and the buffer size was set to $B = 50$.

Applying the tools developed in Sections III–V we can study the chaotic properties of this system. First the Poincaré surface of section is investigated (Fig. 9). Similarly to the three dimensional case, the sum of the congestion windows has to approximately satisfy the $\sum_i w_i = Q$ condition at packet loss times, where Q is the maximum number of packets on the lines and in the buffer. This condition defines the "line of loss" inside the two dimensional window phase space now. The detailed structure of the Poincaré section is shown in the inset in Fig. 9. It can be seen that the Poincaré section is not exactly a line but a narrow grid spreading around the ideal line. This shape can be explained by the packet drop mechanism at the bottleneck buffer. A detailed discussion of this mechanism is given in Section VIII.

To measure the complexity of the TCP dynamics in this situation we apply the symbolic coding that was introduced in Section IV. Then, the topological (Fig. 10) and the Kolmogorov–Sinai (Fig. 11) entropies are estimated. For topological entropy $K_0 = 0.54 \pm 0.01$, while for the KS entropy $K_1 = 0.440 \pm 0.002$ have been obtained. Both values indicate strong mixing and the presence of chaos in the system. However, the maximum number of states is limited due to the discrete phase space, and TCP must enter into a periodic cycle after an initial transient period. If
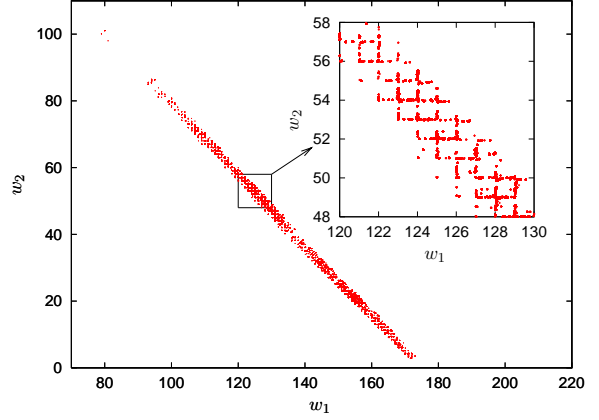

Fig. 9

POINCARÉ SECTION OF THE PHASE SPACE. CONGESTION WINDOW VALUES FOR THE TWO TCPS AT PACKET LOSS TIMES. A SMALL PART OF THE POINCARÉ SECTION IS ZOOMED TO DEMONSTRATE THE FINE STRUCTURE OF THE SECTION.

the orbit realized in the simulation is in fact periodic with some large period, then the number of existing length n sequences $N(n)$ increases only linearly with n if n is sufficiently large. In case of two TCP this linear growth is observable above $n = 15$ (see the inset in Fig. 10). Also the entropy $K_1(n)$ saturates above $n = 15$ and the KS entropy K_1 goes to zero indicating that the long time behavior of the system is periodic.
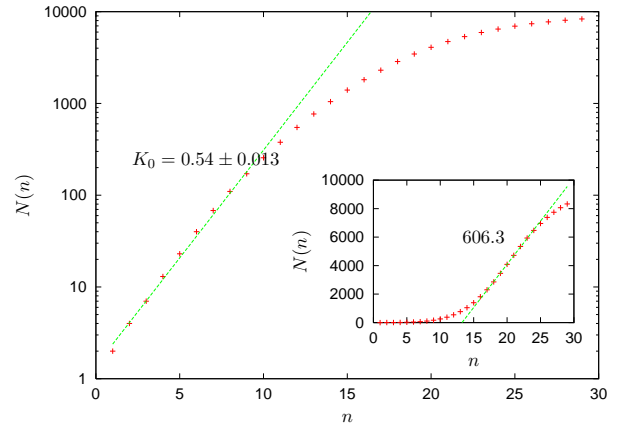

Fig. 10

NUMBER OF DIFFERENT SYMBOLS AS THE FUNCTION OF SYMBOL LENGTH n FOR 2 TCP ON A SEMI-LOGARITHMIC AND ON A NORMAL PLOT IN THE INSET. THE FITTED LINES ARE $N(n) = 1.387 \times 1.718^n$ AND $N(n) = 606.3 \times n - 8034$.

In our 2-TCP scenario periodic orbits were identified. For the given simulation setup the length of the period
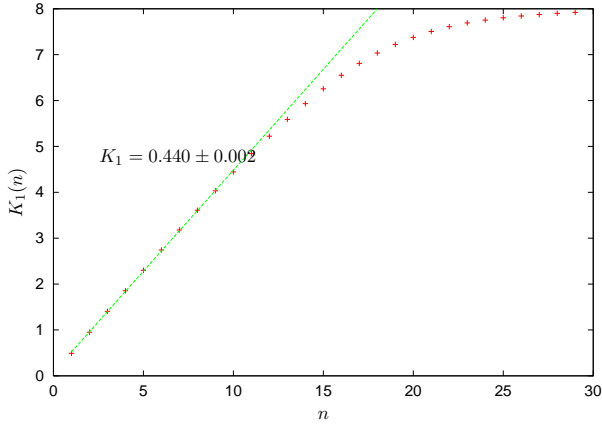
Fig. 11

KOLMOGOROV ENTROPY FOR 2 TCP SCENARIO.
$$K_1(n) = -\sum_{i=1}^{2^n} p_i \ln p_i = 0.440 \times n + 0.081$$

was found to be independent of the initial conditions. The length of the period was $l = 1744$. Using this and Eqs. (9) and (17) we can estimate the order of magnitude of the number of cells in this system. The cells can be indexed by the corresponding symbol sequences. Suppose that we can index uniquely all the cells with symbol sequences of length n^*. In this case the number of cells is approximately given by $e^{K_0 n^*}$. The probability of repeating a cell is then given by

$$\langle p \rangle = \sum_{\{S\}_{n^*}} P^2(S_1, S_2, \dots, S_{n^*}) \approx e^{-K_2 n^*}, \qquad (18)$$

where K_2 is the Rényi entropy for $q = 2$. Combining (17), the estimate for the number of cells $e^{K_0 n^*}$ and (18) one obtains

$$N \approx e^{K_0 n^*} \approx \langle p \rangle^{-K_0/K_2}. \qquad (19)$$

On the other hand (17) implies

$$\langle p \rangle = \frac{\pi}{8} \frac{1}{\bar{l}^2}, \qquad (20)$$

and finally:

$$N \approx \left(\frac{8}{\pi} \bar{l}^2 \right)^{K_0/K_2}. \qquad (21)$$

Assuming that the obtained period $l = 1744$ is a good approximation of the average period and using the values $K_0 = 0.54$ and $K_2 = 0.39$ we get $N \approx 3 \cdot 10^9$. This is in accordance with expectations, since the area in the phase space visited by the congestion window trajectories is approximately a triangle with area $90 \times 90/2$ as one

can see in Fig. 9, while the area of a cell is approximately 0.001^2 in accordance with the findings of Fig. 8. This implies that the visited part of the phase space contains approximately $4 \cdot 10^9$ cells.

VIII. TOWARD CHAOS AWARE MODELING

In the previous sections we managed to collect all the basic elements of chaos in TCP congestion avoidance. Now we would like to show how we can integrate all these into a model without stochastic elements which is able to reproduce all the main features.

The time evolution of congestion windows consists of two parts. The first one is when windows grow steadily. This part is sufficiently well described by existing fluid models like the one given by Eq. (2). In our case just for qualitative comparison we can use a version, where we assume that the buffer is non-empty:

$$\frac{dw_i}{dt} = \frac{1}{T_0 + T_i + bP/C_0}, \qquad (22)$$

where b is the actual number of packets in the buffer. To keep things simple, we estimate the number of packets in the system by the sum of congestion windows and the packets traveling in the lines by $C_0 T_0/P + \frac{1}{3}C_0(T_1 + T_2 + T_3)/P$. This way the queue length in the buffer is estimated as

$$b = \sum_i w_i - C_0 T_0/P + \frac{1}{3}C_0(T_1 + T_2 + T_3)/P.$$

The second and more crucial part of the evolution is the dynamics at packet loss events. In our model we can simplify matters and say that a packet loss occurs whenever the buffer is full and $b = B$ holds. At that time we halve and take the integer part of the congestion window of the TCP whose packet has been lost $w_i' = [w_i/2]$, where $[x]$ denotes the integer part of x.

The elements of the model discussed so far do not generate chaotic dynamics. Now, the really important new element is how we make decision on which TCP loses a packet at a given buffer overflow. In traditional stochastic models this step is random. The argument behind this is that the details of the packet flow are difficult to model on a macroscopic scale and if the packets are well mixed (which is a realistic assumption) then the chance of packet loss for each TCP is proportional with the rate a given TCP sends packets into the buffer. However in reality the packet loss process is more deterministic and predictable than it might seem at first sight. A packet is shifted out of the buffer at each time unit $\tau = P/C$, where P is

the packet size and C is the bandwidth of outgoing packets. When the buffer is full then packet loss will not occur until only one packet comes in within the time slot τ. Packet loss does happen when more than one packets arrive in such a critical time slot. In congestion avoidance mode this can happen if two or more TCP send in packets within the critical slot or if some of the TCPs sends in two packets. Single packets are sent in when a TCP receives an acknowledgement packet, double packets are sent in when in addition the congestion window crosses an integer number. If there are more than one packets sent in during the critical time slot then the one which will make it to the buffer is selected by the exact timing of the packets. This phenomena is called *phase effect* and has been discovered by Floyd and Jacobson in Ref. [15]. Depending on the concrete setup one can determine which TCP will be the winner out of the ones sending packets in the same slot.

In our case the phase analysis shows that the winner TCP is accidentally the one with the largest delay. Consequently this TCP will win all the battles for the critical last buffer space. The only way this TCP can lose packets is when it sends in two packets. This is the reason behind the dominance of this TCP and its very regular behavior observable in Fig. 2(a).

It is clear now that phase effects should be integrated into our model. The difficulty is that the fluid congestion window model is not able to predict when packets are sent out by TCPs. On the other hand, if we start modeling the concrete packet process of TCP then we are back at packet level modeling. So, we have to make a good compromise which saves the simplicity of the fluid equations while keeps the key elements of the packet loss process including phase effects. This can be achieved if we attribute a packet process to the macroscopic fluid evolution of congestion windows. This can be done as follows. Each time the TCP sends out a packet its congestion window is increased by $1/[w]$. Such a way in each round trip time the window increases with 1 and $[w]$ number of packets are sent out. In our simulation we can reverse this and we can calculate the packets sent out by the TCP from the actual macroscopic window changes. We can calculate the fluid window values using the differential equations (22) at the beginning of each time slot:

$$w_i(n) = w_i(t = n \cdot \tau).$$

We can compare $w_i(n + 1)$ and $w_i(n)$ and see if the window crosses in this time slot an integer multiple of $1/[w]$. (This can be done by checking if $[w_i(n + 1)[w_i(n)]]$ and $[w_i(n)[w_i(n)]]$ differ or not.) If it does then we assume that a packet has been sent out by the TCP. If the window

crosses also an integer then we assume that two packets has been sent out in that time slot. This attributed packet process is relevant when the buffer becomes full and the windows reach the loss surface. At this point we should check the sending status of TCPs. If more than one packets arrive during the time slot we let the winner TCP to fill the buffer space while the rest of the TCPs loose packets and halve their windows.

We carried out such a simulation for our model system with three TCP flows. The results are in accordance with the results obtained from the packet level ns simulation. In Fig. 12 we can see that a loss surface similar to Fig. 3 is formed.
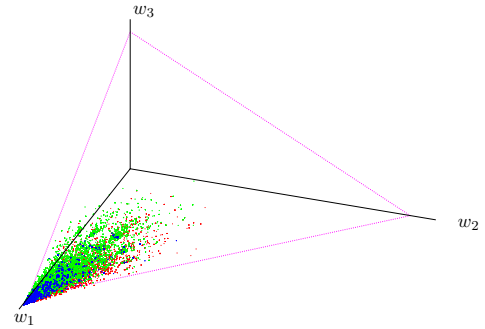

Fig. 12

POINCARÉ SECTION OF THE PHASE SPACE FOR THE MODEL DESCRIBED IN SECTION VIII. CONGESTION WINDOW VALUES FOR THE THREE TCPs AT PACKET LOSS TIMES. COLOURS DEPEND ON WHICH TCP LOST A PACKET. THE DEPICTED (PINK) SURFACE IS WHERE THE SUM OF CONGESTION WINDOWS IS APPROXIMATELY EQUAL WITH THE MAXIMUM NUMBER OF PACKETS WHICH CAN TRAVEL ON THE LINES OR STAY IN THE BUFFER.

The fractal nature of the Poincaré section is also preserved. The fractal dimension of the projection is shown in Fig. 13.

The model creates non-trivial symbolic dynamics similar to the one observed in the ns simulation. The measured topological and KS entropies are shown in Figs. 14 and 15.

This model can be further improved and extended by a more detailed model of the packet loss process including the time slots before the buffer reaches its critical state. The three TCPs are able to send in a maximum of 6 packets during a single time slot and an overflow can happen in a buffer with 5 free packet slots. This effect can cause the broadening of the loss surface as in Fig. 9. Also the time
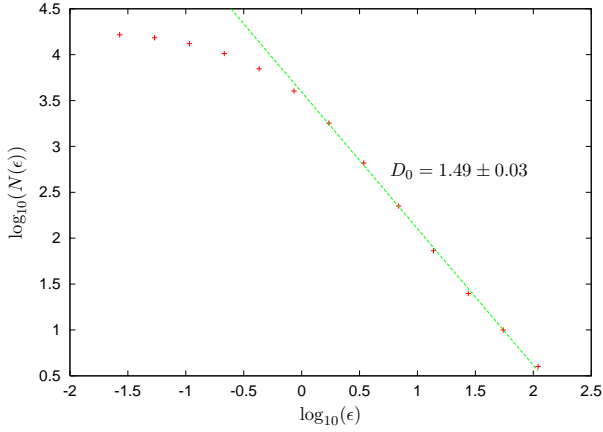
Fig. 13

FRACTAL DIMENSION OF THE PROJECTION OF LOSS EVENTS IN THE PHASE SPACE OBTAINED FROM THE MODEL.
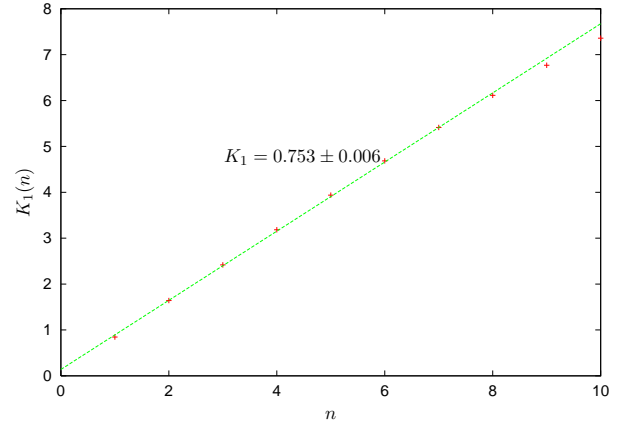

Fig. 15

KOLMOGOROV–SINAI ENTROPY FOR 3 TCP OBTAINED FROM THE MODEL. $K_n = -\sum_{i=1}^{3^n} p_i \ln p_i = 0.753 \times n + 0.108$
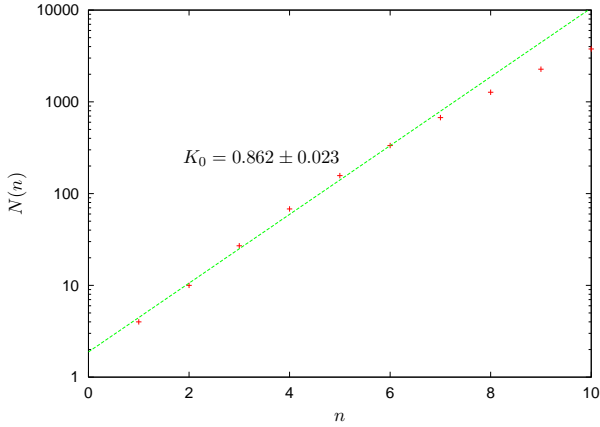

Fig. 14

TOPOLOGICAL ENTROPY FOR 3 TCP OBTAINED FROM THE MODEL.

elapsed between the packet loss event and its notification can be introduced and a more accurate description of the relationship between the round trip time and window values is needed to obtain quantitatively better results.

IX. CONCLUSIONS

In this paper we investigated the chaotic properties of TCPs operating in congestion avoidance mode. We showed that chaotic behavior is general even in the case of low packet loss probability. We demonstrated that the dynamics can be viewed as a smooth time evolution between packet losses and the relevant features of chaos might be described by the investigation of the Poincaré section defined by packet loss events. Chaotic dynamics can be characterized by symbol sequences and we introduced

topological and Kolmogorov–Sinai entropies, whose values confirmed the hypothesis of chaos. Due to the deterministic nature of the system, in contrast to stochastic models, not all possible symbol sequences are realized, some of them are excluded by the dynamics. Accordingly the topological entropy is significantly less than its possible maximal value $\ln L$. The positive Kolmogorov–Sinai entropy indicates that the Lyapunov exponent of the system is positive and that the distribution of symbol sequence probabilities is multifractal. We also proved that due to the cellular structure of the phase space the long time behavior of the system is inherently periodic.

To reproduce these phenomena, we introduced a simple model that incorporates a fluid model of congestion window evolution but also handles packet losses in a deterministic way. In spite of the extensive simplifications we used in the model the basic dynamical characteristics of the TCP dynamics such as the fractal dimension of the attractor and various entropies came out in a qualitatively correct form. The model is also able to count for the apparent unfairness of TCP flows. What we demonstrated here is that instead of a stochastic model of packet loss events, a deterministic model can be worked out and combined with the macroscopic fluid equations. This new model is able to reproduce important phase effects. While the attributed packet generation is not fully identical with the packet generation of a packet level simulation, yet it generates packet flows whose statistical properties are close to the real situation. The model correctly reproduces the loss probabilities of packets arriving in near coincidence into the buffer. This property seems to be essential in generating a proper chaotic time evolution.

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